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THE MOTION OF CROSS-SHAPED BODIES AROUND A FIXED POINT IN A CENTRAL NEWTONIAN FORCE FIELD†

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An octahedral body with identical masses placed at opposite vertices moves around a fixed point in a central field of Newtonian attraction. The body is suspended at its centre of mass, which coincides with its geometrical centre, and the dimensions of the body and the masses concentrated at the vertices are such that all principal central moments of inertia are equal. The problem of whether steady motions of such a body exist is considered, and the stability and bifurcations of certain classes of solutions are investigated. The results are compared with similar results for steady motions of a rigid body whose mass distribution admits of the symmetry group of a regular octahedron [1].

1. We shall consider the motion in a central gravitational field of a rigid body formed by a triple of weightless mutually perpendicular rods l_1, l_2 and l_3 that have exactly one common point—their midpoint O. The lengths of the rods are $2a_1$, $2a_2$ and $2a_3$ and the masses concentrated at the opposite ends of each are respectively m_i, where (and throughout this paper) $i = 1, 2, 3$.

Let $Ox_1x_2x_3$ be a right Cartesian system of coordinates attached to the body with axes directed along the rods l_1, l_2 and l_3, and let C be an attracting centre. In this system of coordinates

$$\overline{CO} = r(\gamma_1, \gamma_2, \gamma_3), \ |\overline{CO}| = r$$

$\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector and $I = \operatorname{diag}(I_1, I_2, I_3)$ is the principal central tensor of inertia. Let us assume that $I_1 = I_2 = I_3 = I$. Then $m_1a_1^2 = m_2a_1^2 = m_3a_3^2$ and the expressions for the kinetic and potential energies are

$$T = \tfrac{1}{2} I \sum_i \omega_i^2, \quad U = -fM \sum_i m_i (1/\rho_i^+ + 1/\rho_i^-)$$

$$\rho_i^{\pm} = (r^2 \pm 2ra_i\gamma_i + a_i^2)^{-1/2}$$

where f is the universal gravitational constant and M is the mass of the attracting centre.

The equations of motion are

$$I\omega^{\cdot} = \gamma \times \partial U / \partial \gamma, \quad \gamma^{\cdot} = \gamma \times \omega$$

They have first integrals: $H = T + U = h$ is the energy integral, $J_1 = I(\omega, \gamma) = p_\psi$ is the integral of the projection of the angular momentum vector onto the γ axis and $J_2 = (\gamma, \gamma) = 1$ is the geometric integral. We lack one more integral for these equations to be integrable in the general case.

2. In order to seek steady motions, let us consider the critical points of the reduced potential energy

$$W = p_\psi^2 (2I(\gamma))^{-1} + U(\gamma), \quad I(\gamma) = \sum_i I_i \gamma_i^2$$

at a level of the geometrical integral fixed in a natural way. Since all the moments of inertia are assumed to be equal, it follows that $I(\gamma) \equiv I$ and the problem of steady solutions and sufficient conditions for their stability is equivalent to the problem of investigating the system's sets of equilibrium positions and their stability.

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Consider the critical points of the function

$$W_\mu = W + \mu((\gamma, \gamma) - 1)/2$$

They are defined by the equations

$$\partial W_\mu / \partial \gamma_i = fMm_i r a_i\left((1/\rho_i^+)^3 - (1/\rho_i^-)^3\right) + \mu\gamma_i = 0, \ \mu = -(\gamma, \partial U / \partial \gamma) \tag{2.1}$$

In Routh's theorem, the sufficient conditions for stability of steady motions are defined as conditions for the quadratic form

$$2\delta^2 W_\mu = \sum_i (\partial^2 U / \partial \gamma_i^2 + \mu)\delta\gamma_i^2$$

to be positive definite on the linear manifold $\delta J_2 = \{\delta\gamma: (\gamma, \delta\gamma) = 0\}$.

There are no terms in the expression for the second variation with $\delta\gamma_i$, $\delta\gamma_j$, $i \neq j$, because the mixed derivatives vanish. The other second derivatives in the expression are

$$\partial^2 U / \partial \gamma_i^2 = -3fMm_i r a_i\left((1/\rho_i^+)^5 + (1/\rho_i^-)^5\right)$$

Equations (2.1) have solutions

$$\gamma_i = \pm 1, \ \gamma_j = 0, \ j \neq i \tag{2.2}$$

In these solutions the ith axis of the body points towards the attracting centre and the body rotates at a constant angular velocity about that axis. For such solutions the linear manifold has the form $\delta J_2 = \{\delta\gamma: \delta\gamma_i = 0\}$, and therefore the motion will be stable if the coefficients of $\delta\gamma_j^2$ ($j \neq i$) are positive. Denoting $\sigma_i = a_i/r$, we write these conditions as follows: ($j \in \{1, 2, 3\}\backslash\{i\}$) (compare [2])

$$f(\sigma_i, \sigma_j) = \frac{3+\sigma_i^2}{(1-\sigma_i^2)^3} - \frac{3}{(1+\sigma_j^2)^{5/2}} > 0, \ \sigma_i < 1 \tag{2.3}$$

$$f(\sigma_i, \sigma_j) = \frac{3\sigma_i^2+1}{(\sigma_i^2-1)^3} - \frac{3\sigma_j}{(1+\sigma_j^2)^{5/2}} > 0, \ \sigma_i > 1 \tag{2.4}$$

Investigation shows that the stability conditions (2.3) are always satisfied, and if $\sigma_i < 1$, the motion under consideration is always stable. The stability condition (2.4) is not always satisfied. Hence, if $\sigma_i > 1$, the set of parameter values in the space $R^3(\sigma_1, \sigma_2, \sigma_3)$ for which the function $f(\sigma_i, \sigma_j)$ vanishes defines bifurcation surfaces on which the degree of instability of the corresponding steady motions varies. These surfaces are cylinders whose generators are parallel to the kth axis, $k = \{1, 2, 3\}\backslash\{i, j\}$. The directrices of the cylinders are defined by the equations $f(\sigma_i, \sigma_j) = 0$, and if $f(\sigma_i(0), 0) = 0$, then $\sigma_i(0) > 1$.

These stability conditions may be compared with the stability conditions of analogous solutions for a regular octahedron. It has been proved [1] that solutions of this kind for a regular octahedron are stable for any ratio a_i/r. The above investigation shows that, subject to the assumptions made about the set of bodies considered, such solutions maintain their stability over a fairly wide range of parameters, which includes the parameters studied in [1].

Let us consider whether there are what are known as "oblique" permanent rotations, when the body is so positioned that none of the x_i axes faces the attracting centre. We first consider the question of whether rotations may exist in which the attracting centre is located in one of the planes Ox_ix_j but not on either of the x_i and x_j axes. To that end we change variables

$$\gamma_i = \sin\vartheta \sin\varphi, \ \gamma_j = \sin\vartheta\cos\varphi, \ \gamma_k = \cos\vartheta$$

and look for permanent rotations by solving the equations

$$\partial W/\partial\varphi = 0, \ \partial W/\partial\vartheta = 0$$

The second equation is satisfied at $\upsilon = \pi/2 + \pi l$, $l = 0, \pm 1, \ldots$. Consider the solution corresponding to the angle $\upsilon = \pi/2$. Then the first equation may be written outside the set $\{\varphi = 0, \varphi = \pi/2 \pmod \pi\}$ as

$$(1+\sigma_j^2)^{-3/2} f_j(\varphi) = (1+\sigma_i^2)^{-3/2} f_i(\varphi) \tag{2.5}$$

$$f_j(\varphi) = F(\varepsilon_j \cos\varphi)/\cos\varphi, \quad f_i(\varphi) = F(\varepsilon_i \sin\varphi)/\sin\varphi$$

$$F(x) = (1-x)^{-3/2} - (1+x)^{3/2}, \quad \varepsilon_i = 2\sigma_i(1+\sigma_i^2)^{-1}$$

Let us investigate the properties of the functions $f_j(\varphi)$ and $f_i(\varphi)$ in the interval $(0, \pi/2)$. Expanding the function f_j in a convergent series in powers of the parameter $\varepsilon_j \cos\varphi \in (0, 1)$, we get

$$f_j(\varphi) = \sum_{l=0}^{\infty} f_{jl}(\varepsilon_j)\cos^{2l}\varphi \tag{2.6}$$

where all the functions $f_{j1}(\varepsilon_j)$ are positive. Consequently, since any natural power of the cosine is a strictly monotone decreasing function in $(0, \pi/2)$, the same is true of $f_j(\varphi)$: it decreases monotonically from $F(\varepsilon_j)$ to $3\varepsilon_j$.

Expanding $f_i(\varphi)$ in a convergent series in powers of the parameter $\eta_i \sin\varphi \in (0, 1)$, we obtain an expression analogous to (2.6) with ε_j replaced by ε_i and $\cos\varphi$ by $\sin\varphi$ with the same coefficients $f_{il} = f_{jl}$ for any l. Since any natural power of the sine is a strictly monotone increasing function in $(0, \pi/2)$, it follows that $f_i(\varphi)$ also increases (strictly) monotonically from $3\varepsilon_i$ to $F(\varepsilon_i)$.

Thus, if the condition

$$(1+\sigma_j^2)^{-3/2} F(\varepsilon_j) > 3(1+\sigma_i^2)^{-3/2}\varepsilon_i$$

and the analogous condition with the subscripts i and j interchanged are both satisfied, then Eq. (2.4) has a unique solution as a function of φ.

Comparing with the conditions for the varying degree of instability of the "direct" solutions considered hitherto, one sees that the set of oblique solutions in the space $R^3(\sigma_1, \sigma_2, \sigma_3) \times S^2(\gamma_1, \gamma_2, \gamma_3)$ branches off from one of the direct solutions $\{\gamma_i = \pm 1, \gamma_j = 0, j \neq i\}$ as its degree of instability varies with the parameters. This set tends to another such solution as the parameter values tend to the values at which the degree of instability of those other solutions varies. This approach to the branching of steady motions may also be extended to the problem of the orbital dynamics of rigid bodies of octahedral shape made up, for example, of homogeneous rods or of mass-less rods with equal masses at their opposite ends [2–4].

3. The investigation of the sufficient conditions for the stability of oblique solutions, in the general case, is rather difficult, as these solutions are not available as explicit functions of the parameters. Nevertheless, if the conditions $a_i = a_j = a$ holds, oblique solutions may be determined explicitly. They have the form

$$\gamma_i = \pm\gamma_j = 1/\sqrt{2}, \quad \gamma_k = 0 \tag{3.1}$$

and one can write down explicit stability conditions for them. For example, for the solution

$$\gamma_i = \gamma_j = 1/\sqrt{2}, \quad \gamma_k = 0$$

the linear manifold is

$$\delta J_2 = \{\delta\gamma: \delta\gamma_i + \delta\gamma_j = 0\}$$

and the stability conditions reduce to conditions for the quadratic form

$$2\delta^2 W_\mu = (\partial^2 U/\partial\gamma_i^2 + \partial^2 U/\partial\gamma_j^2 + 2\mu)\delta\gamma^2 + (\partial^2 U/\partial\gamma_k^2 + \mu)\delta\gamma_k^2$$

$$\delta\gamma = \delta\gamma_i = -\delta\gamma_j$$

to be positive definite. The condition for the first coefficient to be positive may be expressed, apart from a positive factor, as

$$-3\sigma(\Delta_+^{-5}+\Delta_-^{-5})+\sqrt{2}(\Delta_-^{-5}-\Delta_+^{-5})>0 \tag{3.2}$$

$$\sigma=a/r,\ \Delta_\pm=\left(1\pm\sqrt{2}\sigma+\sigma^2\right)^{1/2}$$

This condition never holds. Consequently, the solution has a degree of instability of at least one. The condition for the second coefficient to be positive is

$$g(\sigma,\ \sigma_k)=(\Delta_-^{-3}-\Delta_+^{-3})\sigma^{-1}-3\sqrt{2}(1+\sigma_k^2)^{5/2}>0 \tag{3.3}$$

The domain in the (σ, σ_k) plane in which condition (3.3) holds lies to the right of the set $g(\sigma, \sigma_k) = 0$, and if $g(\sigma(0), \sigma_k) = 0$, then $0 < \sigma(0) < 1$.

Let us compare these conditions with the conditions for the stability of the analogous solutions for a regular octahedron. It has been proved [1] that, for a regular octahedron, solutions of this type have a degree of instability of one for any ratio a_i/r. Our investigation shows that, subject to the assumptions made about the set of bodies under consideration, the degree of instability of these solutions is preserved over a fairly large range of parameters, including those studied in [1].

4. Let us investigate whether steady motions exist such that the axis of rotation of the body lies inside one of the octants. We will again assume that $a_1 = a_2$. Then, introducing angles θ and φ, as previously, we can seek such solutions, say, in the set $\varphi = \pi/4$. In this situation, the desired solution must satisfy the following equation in the interval $\theta \in (0, \pi/2)$

$$m(1+\sigma^2)^{-1/2}\varepsilon f(\theta)=m_3(1+\sigma_k^2)^{-1/2}2\varepsilon_3 f_3(\theta)$$

$$f_3(\upsilon)=F(\varepsilon_3\cos\vartheta)/\cos\vartheta,\ \ f(\upsilon)=F(\varepsilon\sin\vartheta/\sqrt{2})/(\sin\vartheta/\sqrt{2})$$

$$\varepsilon = 2\sigma\ (1+\sigma^2)^{-1}$$

Continuing the investigation as in Section 2, we conclude that solutions with the axis of rotation in the first octant exist provided that

$$m_3(1+\sigma_3^2)^{-1/2}\varepsilon_3F(\varepsilon_3)>3m(1+\sigma^2)^{-1/2}\varepsilon^2$$

$$m(1+\sigma^2)^{-1/2}(\varepsilon/\sqrt{2})F(\varepsilon/\sqrt{2})>3m_k(1+\sigma_k^2)^{-1/2}\varepsilon_k^2/2$$

This class of solutions includes the rotation of a regular octahedron, with identical masses at the vertices, about an axis passing through the centres of two faces, which was studied in [1].

5. We will now consider the integrability of the approximate equations of motion of the mechanical system under consideration, but assuming the moments of inertia to be arbitrary. We know that the approximate equations of motion obtained by expanding the potential in terms of small parameters of the same type as σ_i, up to second-order terms, are completely integrable. At the same time:

1. if all three principal central moments of inertia of the body coincide, the approximate equations of motion are the equations of inertial motion of a homogeneous sphere and the dynamics of the body is trivial;
2. if only two of the principal central moments of inertia coincide, the equations of motion admit of another first-degree integral, analogous to the additional Kirchhoff integral in the problem of a body moving in a fluid;
3. if the three principal central moments of inertia are all different, the equations of motion admit of another integral, analogous to the integral in Clebach's first case in the problem of a body moving in a fluid.

Let us consider the approximate equations obtained by expanding the potential in powers of the parameters ε_i up to second-order terms. We have

$$U=U_0+U_2+\ldots=-2fM\sum_i m_i\left(r^2+a_i^2\right)^{-1/2}+\frac{1}{2}\sum_i c_i\gamma_i^2\ldots$$

$$c_i=-6fMr^2m_ia_i^2(r+a_i^2)^{-5/2},\ \ i=1,\ 2,\ 3$$

If the principal central moments of inertia are equal, the approximate equations of motion of this rigid body about a fixed point are completely integrable: their integral is identical with the first integral in Clebach's "second case" of the problem of a rigid body moving in an infinite volume of an incompressible fluid. The additional integral is

$$J = \frac{1}{2}\sum_i c_i\omega_i^2 - \frac{1}{2}(c_2c_3\gamma_1^2 + c_3c_1\gamma_2^2 + c_1c_2\gamma_3^2)$$

and moreover, unlike the case in which the expansions were done in terms of the parameters σ_i, the dynamics described by these approximate equations are by no means trivial: the motion is described in terms of θ-functions of time.

If only two of the principal central moments of inertia are equal, say $I_1 = I_2$, one can indicate only two cases in which the equations of motion admit of an additional integral. In one of those cases we have $c_1 = c_2$ (so that $m_1 = m_2, a_1 = a_2$), and the additional integral is analogous to the Kirchhoff integral in the motion of a body in a fluid, of the form $J = \omega_3$. In the other case, which occurs only at the zero level of the area integral ($J_1 = 0$), we have $I_1 = I_2 = 2I_3$ (so that $m_1a_1^2 = m_2a_2^2 = m_3a_3^2/3$), and $c_1 + c_2 = 2c_3$. Here the additional integral is analogous to Chaplygin's particular integral in the problem of a body moving in a fluid, which is [5]

$$J = \left(I_3\omega_1^2 - I_3\omega_2^2 + c\gamma_3^2\right)^2 + 4\omega_1^2\omega_2^2I_3^2, \quad c_1 - c_2 = c_2 - c_3 = 2c$$

and the equations are integrated in terms of elliptic functions.

As in the case of the motion of a body in a liquid [6] (see also [7]), the equations of motion in the case when two principal moments of inertia are equal admit of no other cases in which an additional general first integral exists, apart from those listed above.

Finally, if the three principal central moments of inertia are different, the approximate equations of motion are not always integrable, as happens when one uses small parameters of type σ_i, but only when Clebsch's condition holds [8]

$$I_1(c_2 - c_3) + I_2(c_3 - c_1) + I_3(c_1 - c_2) = 0 \tag{5.1}$$

or $c_i = -\nu I_i + \mu$, which is analogous to Clebsch's condition in the motion of a rigid body in an ideal fluid. Under these conditions the additional integral is

$$J = \sum_i I_i^2\omega_i^2 + \nu(I_2I_3\gamma_1^2 + I_3I_1\gamma_2^2 + I_1I_2\gamma_3^2)$$

6. An integrable case of the equations of motion may be pointed out in a more complicated formulation of the problem. Let $OX_1X_2X_3$ be an absolute system of coordinates and let C_i be attracting centres on the X_i axes, respectively, with

$$\overline{C_iO} = r_i \cdot \mathbf{S}_i, \quad |\overline{C_iO}| = r_i, \quad \mathbf{S}_i = (S_{i1}, S_{i2}, S_{i3})$$

where M_i are masses concentrated at the points C_i.

The equations of motion are

$$I\omega^{\cdot} = I\omega\times\omega + \sum_i \mathbf{S}_i \times \partial U/\partial\mathbf{S}_i, \quad \mathbf{S}_i^{\cdot} = \mathbf{S}_i \times \omega \tag{6.1}$$

These equations, apart from the energy integral, have six geometrical integrals reflecting the orthonormality of the basis $\mathbf{S}_i$. For these equations to be integrable in the general case, a further two first integrals are needed.

We introduce the parameters

$$\varepsilon_{ij} = 2r_ia_j(r_i^2 + a_j^2)^{-1}$$

and expand the potential in power of φ_{ij} up to second-order terms. Then

$$U = U_0 + U_2 + \ldots = -2f\sum_i M_i \sum_j m_j(r_i^2 + a_j^2)^{-1/2} + \frac{1}{2}\sum_i\sum_j c_{ij}S_{ij}^2 + \ldots$$

$$c_{ij} = -\frac{3}{2}M_im_j(r_i^2 + a_j^2)^{-5/2}(2r_ia_j)^2 \quad i,\ j = 1,\ 2,\ 3$$

and we can consider the existence of additional integrals of the approximate equations of motion.

We shall consider two extreme cases.

Suppose that all the principal central moments of inertia are equal. Then, if

$$c_{ij} = -x_i r_j + y_i, \quad i, j = 1, 2, 3$$

the approximate equations of motion are completely integrable. Under these conditions the additional integrals are quadratic with respect to each of the variables and have the form

$$J_1 = I(\omega_1^2 + \omega_2^2 + \omega_3^2) + \sum_i x_i \sum_j r_j S_{ij}^2$$

$$J_2 = -I \sum_i x_i (M \times S_i)^2 + \sum_i x_i^2 \sum_j r_j S_{ij}^2$$

But if all the moments of inertia are distinct, there is not always a pair of additional integrals, as happens when one uses expansions in terms of the parameters a_i/r_j (see, for example, [4]), but only when certain conditions, analogous to Clebsch's conditions, are satisfied

$$I_1(c_{i2} - c_{i3}) + I_2(c_{i3} - c_{i1}) + I_3(c_{i1} - c_{i2}) = 0, \quad i = 1, 2, 3$$

Thus, the integrability of the approximate equations of motion depends not only on equality of the inertial and gravitational masses, as pointed out in [9, p. 25], but also on how the small parameters are defined in the problem.

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